PERGAMON

Applied Mathematics Letters 12 (1999) 33–40

Applied Mathematics Letters

# Analysis by Fuzzy Difference Equations of a Model of $CO_2$ Level in the Blood

E. Y. DEEBA AND A. DE KORVIN
Department of Computer and Mathematical Sciences
University of Houston-Downtown
One Main Street, Houston, TX 77002, U.S.A.
<deebae@dt.uh.edu> <deKorvin@dt.uh.edu>

(Received February 1998; revised and accepted May 1998)

**Abstract**—In this paper we shall consider a model to determine the carbon dioxide ($CO_2$) level in the blood. The model consists of a set of nonlinear difference equations. However, the linearized model will be solved. Since many measurements and factors that determine the $CO_2$ level in the blood may be imprecise, we will consider the fuzzy analog of the linearized model as a method to compensate for these imprecise measurements. We will estimate, for a fixed threshold $\alpha$, a solution to the fuzzy difference equation with belief at least $\alpha$. We will show that the results reduce to the classical case when the fuzzy quantities are replaced by crisp ones. 

**Keywords**—Fuzzy logic, Difference equations, Modeling.

## 1. INTRODUCTION

Difference Equations arise in the modeling of many interesting phenomena such as in population dynamics, random walks, probability models for learning, economics, and approximate solutions of ordinary and partial differential equations just to mention a few (see [1–5]). We refer the reader to [1,4,5] for a detailed study of the Theory of Difference Equations.

In this paper, we shall consider a model for determining the carbon dioxide ($CO_2$) in the blood. The normal range of $CO_2$ in the blood is 24–30 ml/liter for adults and 20–26 ml/liter for infants. The level at time $n+1$, depends upon several factors including the concentration of $CO_2$ present at time $n$, ventilation rate (the number of times a person ventilates per minute), tidal volume (the actual volume of air that moves into, then out of the lungs with each breath) at time $n+1$, metabolism, and other physiological factors. The mathematical model that describes the level of $CO_2$ is summarized below (see [2]). Let $C_n$ be the concentration of $CO_2$ in the blood cycle $n$ and $V_n$ be the pulmonary ventilation (the product of the tidal volume and the ventilation rate) in cycle $n$. The concentration of $CO_2$ lost in cycle $n$ is a nonlinear function $L(C_n, V_n)$. Let $F$ be a nonlinear function relating $V_n$ and $C_n$. Further, let $m$ be the concentration of $CO_2$ in the

This research was partially supported by grants from NSF (CDA-9522157) and the ARO (DAAH-0495-10250).

0893-9659/99/$ - see front matter  Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(98)00168-2

blood due to metabolism. The mathematical model that expresses the level of $CO_2$ is:

$$C_{n+1} = C_n - L(C_n, V_n) + m \tag{1.1}$$

$$V_{n+1} = F(C_n). \tag{1.2}$$

Equations (1.1) and (1.2) represent a nonlinear model that determines the concentration of $CO_2$ in the blood in the $(n+1)$ cycle. Indeed, the consideration of some basic physiological features of $CO_2$ transport in the organism lead to the above model. We will simplify this model by assuming that

(i) the concentration of $CO_2$ lost, $L(C_n, V_n)$, does not depend on $C_n$ and is directly proportional to the ventilation volume $V_n$, with constant of proportionality $a \in (0,1)$, that is, $L(C_n, V_n) = aV_n$, and

(ii) the ventilation volume in the $(n+1)$ cycle is directly proportional to $C_n$, that is, equation (1.2) reads $V_{n+1} = bC_n$ with constant of proportionality $b \in (0,1)$.

Under these assumptions equations (1.1) and (1.2) reduce to the linear model for determining the concentration of $CO_2$ in the blood

$$C_{n+1} = C_n - abC_{n-1} + m. \tag{1.3}$$

The metabolism measurements as well as the proportionality assumption stated above are inherently imprecise. This motivates us to consider the fuzzy analog of this linearized model. In this paper, we shall study and analyze the behavior of equation (1.3) and then consider it in the fuzzy setting. The fuzzy analog of (1.3) will result in a coupled system of difference equations. We will solve the coupled system and analyze the behavior of the behavior of the solution. Indeed, we will show that if $ab$ is not too "uncertain", then a solution to the fuzzy equation will be determined. In the next section we will give the background material needed to cast equation (1.3) in the fuzzy setting.

## 2. PRELIMINARIES

We present, for the sake of completeness, some background material needed in this sequel. For a detailed study, we refer the reader to [6–13]. Let $X$ be any nonempty set. A set $A$ is said to be a *fuzzy* set on $X$ if $A$ is a function from $X$ into the interval [0,1]. The value $A(x)$ is referred to as the membership of $x$ in $A$. $R_t^+$ denotes the set of all reals greater or equal to a positive number $t$. Let $A$ be a fuzzy subset of $R_t^+$. This implies that the support of $A$ is disjoint from $[0,a]$ for $a < t$; that is, $A(x) = 0$ if $x \in [0,a]$. For example, let $A$ represent the age (in years) of a middle-aged person and let $t = 5$. We can safely assert that $A$ is a subset of $R_5^+$; that is, any age between 0 and 5 has membership 0 in $A$. $A$ is said to be *convex* if for every $s \in [0,1]$,

$$A(sx_1 + (1-s)x_2) \geq \min\{A(x_1), A(x_2)\}.$$

$A$ is said to be *normalized* if there exists an $x$ such that $A(x) = 1$. The $\alpha$-level of a fuzzy subset $A$, denoted by $[A]_\alpha$, is defined by

$$[A]_\alpha = \{x \in R_t^+ \mid A(x) \geq \alpha\}.$$

It is well known [7] that the $\alpha$-levels of a fuzzy subset $A$ determine $A$. Also, it can be easily verified that for any fuzzy subset $A$ of $R_t^+$:

(C1) $A$ is convex if and only if $[A]_\alpha$ is convex, and

(C2) if $A$ is continuous, then $A$ is convex if and only if $[A]_\alpha$ is a closed interval.

By a *fuzzy number* $N$, we mean a fuzzy subset of $R_t^+$ which is continuous, convex, normalized, and vanishing at infinity.

Let $P$ denote the product $R_t^+ \times R_t^+ \times R_t^+ \times R_t^+ \times R_t^+$. Let $g$ be a function defined $g : P \to R$. The Extension Principle (see [7,10–13]) states that $g$ can be extended to five tuples $(A, B, C, D, E)$ where $A, B, C, D$, and $E$ are fuzzy subsets of $R_t^+$ as follows:

$$g(A,B,C,D,E)(y) = \sup\left(\min\left\{A(y_1), B(y_2), C(y_3), D(y_4), E(y_5)\right\}\right), \tag{2.1}$$

where the sup is taken over all $y_1$, $y_2$, $y_3$, $y_4$, and $y_5$ such that $g(y_1, y_2, y_3, y_4, y_5) = y$.

Finally, we will need a basic result of Nguyen [9] which addresses the following question. When does the relation

$$[g(A,B,C,D,E)]_\alpha = g\left([A]_\alpha, [B]_\alpha, [C]_\alpha, [D]_\alpha, [E]_\alpha\right) \tag{2.2}$$

hold for any continuous function $g$ and for any fuzzy subsets $A, B, C, D, E$? It was shown in [9] that (2.2) holds if and only if the sup in (2.1) is attained; that is, there exist $y_1^*$, $y_2^*$, $y_3^*$, $y_4^*$, and $y_5^*$ such that

$$g(A,B,C,D,E)(y) = \min\left\{A(y_1^*), B(y_2^*), C(y_3^*), D(y_4^*), E(y_5^*)\right\}, \tag{2.3}$$

where $y = g(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*)$.

## 3. THE CLASSICAL DIFFERENCE EQUATION

In this section we will consider the equation

$$C_{n+1} = C_n - abC_{n-1} + m. \tag{3.1}$$

Assume that the solution to the homogeneous equation

$$C_{n+1} = C_n - abC_{n-1}$$

is $\lambda^n$. This yields the characteristic equation $\lambda^2 - \lambda + ab = 0$. Thus the solution of equation (3.1) is given by

$$C_n = c_1\left(\frac{1+\sqrt{1-4ab}}{2}\right)^n + c_2\left(\frac{1-\sqrt{1-4ab}}{2}\right)^n + \frac{m}{ab}, \tag{3.2}$$

where $c_1$ and $c_2$ are arbitrary constants and can be determined upon specifying $C_0$ and $C_1$. The behavior of the solution depends upon the quantity $1 - 4ab$. We consider three cases.

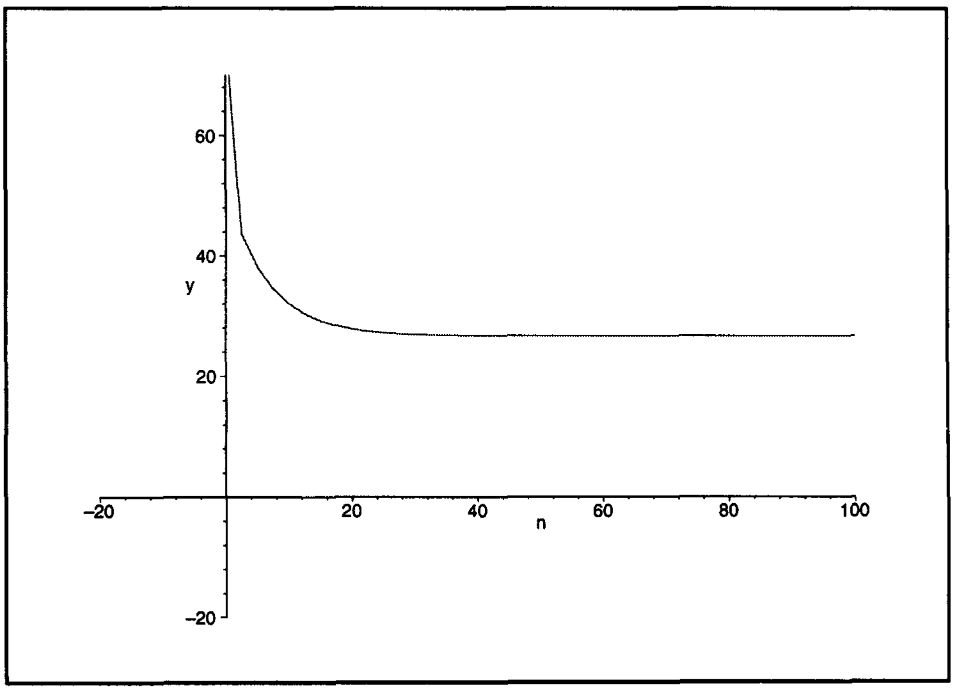


Figure 1.

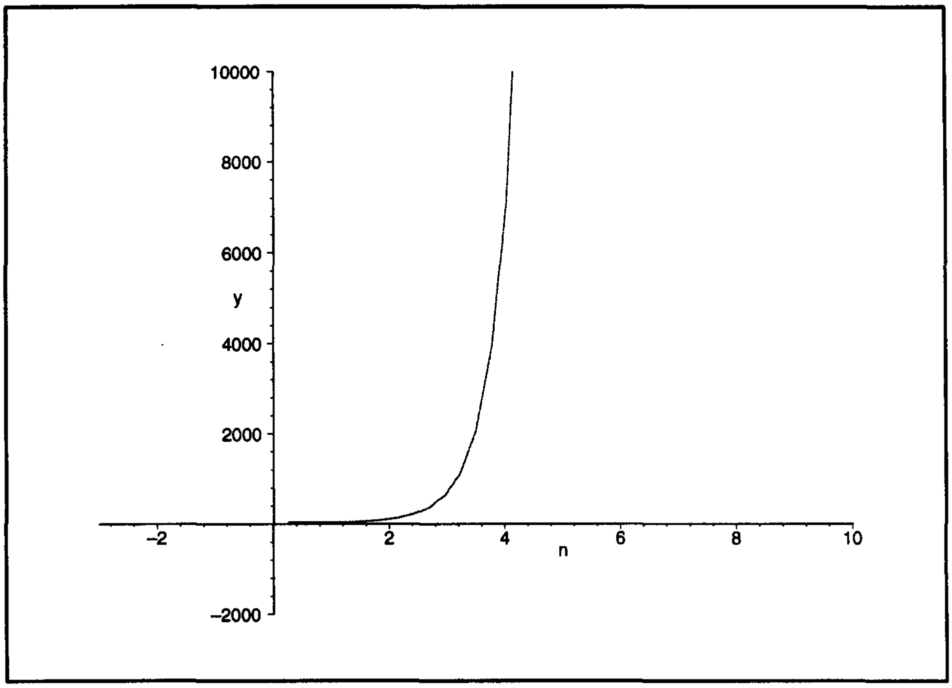


Figure 2.

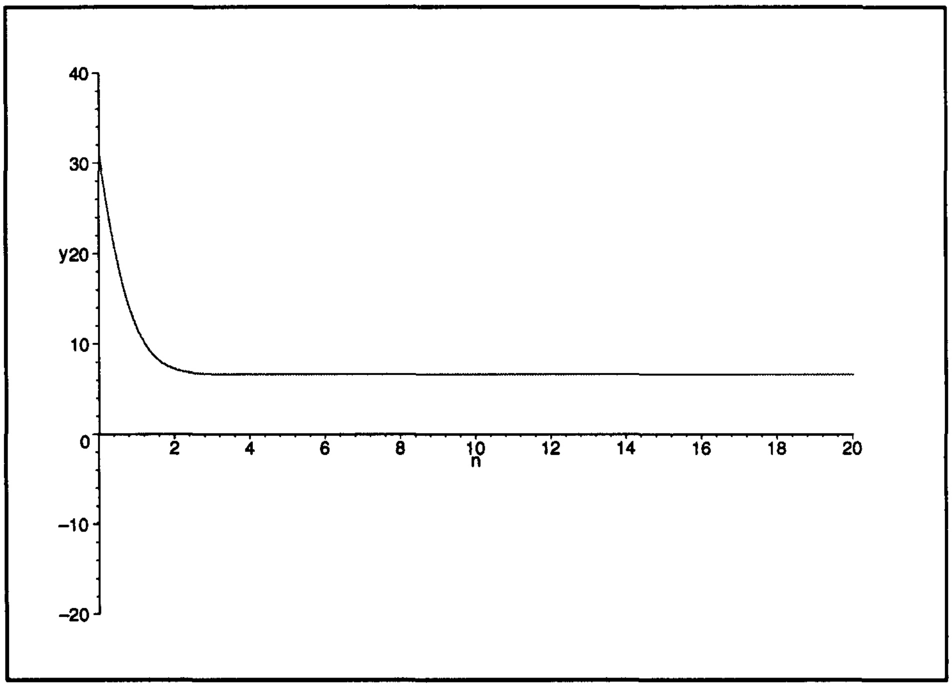


Figure 3.

CASE 1. In this case we assume that $1-4ab > 0$. The solution will approach $m/ab$ as $n$ increases regardless of the initial conditions. For example, for $a = 0.3$, $b = 0.4$, and $m = 3.2$, the solution will approach 26.67 units/liter after 100 iterations. If $C_0 = 24.5$ and $C_1 = 26$, the concentration of $CO_2$ in the blood is graphically represented as in Figure 1.

CASE 2. In this case we assume that $1 - 4ab = 0$. The solution will not be given by (3.2). The roots of the characteristic equation are equal and we seek two independent solutions. In this case, the concentration of $CO_2$ will grow exponentially without a limit. For example, if $a = 0.5$ and

$b = 0.5$ and $C_0 = 24.5$ and $C_1 = 26$, then the concentration of $CO_2$ in the blood is graphically represented as in Figure 2.

CASE 3. In this case we assume that $1 - 4ab < 0$. Figure 3 is the graph of the solution with $a = 0.7$, $b = 0.6$, $C_0 = 24.5$, $C_1 = 26$, and $m = 3.2$. The concentration of $CO_2$ oscillates and will reach the steady state solution $m/ab$.

## 4. THE FUZZY ANALOG OF THE DIFFERENCE EQUATION (3.1)

In this section we shall consider the difference equation

$$C_{n+1} = C_n - abC_{n-1} + m \tag{4.1}$$

with initial conditions $C_0$ and $C_1$.

Equation (4.1) is the "classical" difference equation where all the variables and parameters involved are real numbers. The solution is given by (3.2). By the extension theorem (see, [7,10–13], the right-hand side of (4.1) means

$$(C_{n+1})(y) = \sup\left(\min\left\{C_n(y_1), C_{n-1}(y_2), a(y_3), b(y_4), m(y_5)\right\}\right), \tag{4.2}$$

where the sup is taken over all $y_1$, $y_2$, $y_3$, $y_4$, and $y_5$ for which $y = f(y_1, y_2, y_3, y_4, y_5) = y_1 - y_2y_3y_4 + y_5$.

The main purpose of the present work is to study the solution of the fuzzified extension (4.2) of (4.1). Such an extension is naturally generated if we set up (4.1) with reasonable guesses as to what the real values $a, b$, and $m$ could be as well as guessing the initial conditions $C_0$ and $C_1$. The present work is then to assess the impact of fuzzifying parameters and initial conditions on the solution of (4.1).

LEMMA 4.0. *For each fixed $y$, the function $f : P \to R$ is continuous and increasing in its arguments $y_1$ and $y_5$ and decreasing in $y_2$, $y_3$, and $y_4$.*

Since all the variables $y_1$, $y_2$, $y_3$, $y_4$, and $y_5$ are bounded, the next lemma follows.

LEMMA 4.1. *The set $f^{-1}(y) = \{(y_1, y_2, y_3, y_4, y_5) \mid y = f(y_1, y_2, y_3, y_4, y_5)\}$ is a compact subset of $P$.*

LEMMA 4.2. *For all $\alpha \in (0,1]$ and $f$ continuous, we have $[f(A,B,C,D,E)]_\alpha = f([A]_\alpha, [B]_\alpha, [C]_\alpha, [D]_\alpha, [E]_\alpha)$ for any fuzzy numbers $A, B, C, D$, and $E$.*

PROOF. Let $f^{-1}(y) = \{(y_1, y_2, y_3, y_4, y_5) \mid y = f(y_1, y_2, y_3, y_4, y_5)\}$. For any fixed $y$, the set $f^{-1}(y)$ is compact. Denote by

$$b = \sup_{(y_1,y_2,y_3,y_4,y_5)\in f^{-1}(y)} \left(\min\left\{A(y_1), B(y_2), C(y_3), D(y_4), E(y_5)\right\}\right).$$

If $b > 0$, then there is an $M$ such that $\min\{A(y_1), B(y_2), C(y_3), D(y_4), E(y_5)\} < b$ for all $y_i > M (i = 1,2,3,4,5)$ since $A, B, C, D$, and $E$ are fuzzy numbers that vanish at infinity. Hence it is sufficient to take the supremum over the compact set $f^{-1}(y) \cap ([0,M] \times [0,M] \times [0,M] \times [0,M] \times [0,M])$. Since $A$, $B$, $C$, $D$, and $E$ are continuous, the supremum is attained. The result follows [9].

LEMMA 4.3. *For all $n$, $[C_{n+1}]_\alpha = [C_n]_\alpha - [a_n]_\alpha[b_n]_\alpha[C_{n-1}]_\alpha + [m]_\alpha$ for any fuzzy numbers $C_n$, $a$, $b$, $C_{n-1}$, and $m$.*

PROOF. Setting $f$ to be $f(y_1, y_2, y_3, y_4, y_5) = y_1 - y_2y_3y_4 + y_5$ and since $f$ is continuous, the result follows immediately from Lemma 4.2.

LEMMA 4.4. *If $C_n$ is a fuzzy number, then the $\alpha$-level $[C_n]_\alpha$ is a closed bounded subinterval of $R_t^+$. Likewise, $[a]_\alpha$, $[b]_\alpha$, $[C_{n-1}]_\alpha$, and $[m]_\alpha$ are closed bounded intervals.*

PROOF. Since $C_n$ is a fuzzy number, it follows that $[C_n]_\alpha$ is convex. By continuity, $[C_n]_\alpha$ is closed in $R_{n,\alpha}$ and is bounded since $\lim_{x\to\infty} C_n(x) = 0$. Thus, $[C_n]_\alpha$ is a closed bounded interval of $R_t^+$.

Lemma 4.4 implies the following.

LEMMA 4.5. *If $C_n$ is a fuzzy number, then the $\alpha$-levels $[C_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}]$, where $L_{n,\alpha}$ and $R_{n,\alpha}$ denote, respectively, the left and the right endpoints of the interval. Likewise, $[a]_\alpha = [a_{l,\alpha}, a_{r,\alpha}]$, $[b]_\alpha = [b_{l,\alpha}, b_{r,\alpha}]$, $[C_{n-1}]_\alpha = [L_{n-1,\alpha}, R_{n-1,\alpha}]$, and $[m]_\alpha = [m_{l,\alpha}, m_{r,\alpha}]$.*

Lemma 4.3 together with Lemma 4.5 imply that

$$[L_{n+1,\alpha}, R_{n+1,\alpha}] = [L_{n,\alpha}, R_{n,\alpha}] - [a_{l,\alpha}, a_{r,\alpha}]\,[b_{l,\alpha}, b_{r,\alpha}]\,[L_{n-1,\alpha}, R_{n-1,\alpha}] + [m_{l,\alpha}, m_{r,\alpha}]\,.$$

LEMMA 4.6. *The left and the right endpoints of $[C_n]_\alpha$ satisfy the following difference equations:*

$$L_{n+1,\alpha} = L_{n,\alpha} - a_{r,\alpha} b_{r,\alpha} R_{n-1,\alpha} + m_{l,\alpha} \tag{4.3}$$

$$R_{n-1,\alpha} = R_{n,\alpha} - a_{l,\alpha} b_{l,\alpha} L_{n-1,\alpha} + m_{r,\alpha}. \tag{4.4}$$

The system in (4.3) and (4.4) is a coupled system of difference equation. The matrix representation of this system is

$$\begin{bmatrix} L_{n+1,\alpha} \\ R_{n+1,\alpha} \end{bmatrix} = \begin{bmatrix} L_{n,\alpha} \\ R_{n,\alpha} \end{bmatrix} - \begin{bmatrix} 0 & a_{r,\alpha} b_{r,\alpha} \\ a_{l,\alpha} b_{l,\alpha} & 0 \end{bmatrix} \begin{bmatrix} L_{n-1,\alpha} \\ R_{n-1,\alpha} \end{bmatrix} + \begin{bmatrix} m_{l,\alpha} \\ m_{r,\alpha} \end{bmatrix}. \tag{4.5}$$

Set $Y_n = \begin{bmatrix} L_{n,\alpha} \\ R_{n,\alpha} \end{bmatrix}$. Then equation (4.5) is

$$Y_{n+1} = Y_n - AY_{n-1} + M. \tag{4.6}$$

Since the matrix $A = \begin{bmatrix} 0 & a_{r,\alpha} b_{r,\alpha} \\ a_{l,\alpha} b_{l,\alpha} & 0 \end{bmatrix}$ has distinct eigenvalues, it follows that the matrix $A$ is diagonalizable; that is, there is a nonsingular matrix $P$ such that $P^{-1}AP = D$, where $D$ is the diagonal matrix with diagonal elements the eigenvalues of the matrix $A$. These matrices are given by

$$P = \begin{bmatrix} \dfrac{\sqrt{a_{r,\alpha} b_{r,\alpha}}}{\sqrt{a_{l,\alpha} b_{l,\alpha}}} & -\dfrac{\sqrt{a_{r,\alpha} b_{r,\alpha}}}{\sqrt{a_{l,\alpha} b_{l,\alpha}}} \\ 1 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} \dfrac{1}{2} \dfrac{\sqrt{a_{l,\alpha} b_{l,\alpha}}}{\sqrt{a_{r,\alpha} b_{r,\alpha}}} & \dfrac{1}{2} \\ -\dfrac{1}{2} \dfrac{\sqrt{a_{l,\alpha} b_{l,\alpha}}}{\sqrt{a_{r,\alpha} b_{r,\alpha}}} & \dfrac{1}{2} \end{bmatrix}$$

$$D = \begin{bmatrix} \sqrt{a_{r,\alpha} b_{r,\alpha} a_{l,\alpha} b_{l,\alpha}} & 0 \\ 0 & -\sqrt{a_{r,\alpha} b_{r,\alpha} a_{l,\alpha} b_{l,\alpha}} \end{bmatrix}.$$

Multiply equation (4.6) by $P^{-1}$ and set $Z_n = P^{-1}Y_n$ to obtain

$$Z_{n+1} = Z_n - DZ_{n-1} + P^{-1}M. \tag{4.7}$$

If $Z_n = \begin{bmatrix} Z_{1,n} \\ Z_{2,n} \end{bmatrix}$, then equation (4.7) yields two decoupled equations in $Z_{1,n}$ and $Z_{2,n}$. In particular,

$$Z_{1,n+1} = Z_{1,n} - \sqrt{a_{r,\alpha} b_{r,\alpha}, a_{l,\alpha} b_{l,\alpha}}\, Z_{1,n-1} + \frac{1}{2} \frac{\sqrt{a_{l,\alpha} b_{l,\alpha}}}{\sqrt{a_{r,\alpha} b_{r,\alpha}}} m_{l,\alpha} + \frac{1}{2} m_{r,\alpha} \tag{4.8}$$

$$Z_{2,n+1} = Z_{2,n} + \sqrt{a_{r,\alpha} b_{r,\alpha}, a_{l,\alpha} b_{l,\alpha}}\, Z_{2,n-1} - \frac{1}{2} \frac{\sqrt{a_{l,\alpha} b_{l,\alpha}}}{\sqrt{a_{r,\alpha} b_{r,\alpha}}} m_{l,\alpha} + \frac{1}{2} m_{r,\alpha}. \tag{4.9}$$

It is clear that equation (4.8) is of the same form as (3.1) and hence, for each $\alpha$-level,

$$Z_{1,\infty} \to \frac{(1/2)\left(\sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}}\right)m_{l,\alpha} + (1/2)m_{r,\alpha}}{\sqrt{a_{r,\alpha}b_{r,\alpha}a_{l,\alpha}b_{l,\alpha}}} \tag{4.10}$$

if $1-4\sqrt{a_{r,\alpha}b_{r,\alpha}a_{l,\alpha}b_{l,\alpha}} > 0$ which is the case we analyze below then the above corresponds to the classical case with two distinct real roots. On the other hand, the quantity $Z_{2,n}$ diverges. Thus we cannot compute the left and right endpoints of $[C_n]_\alpha$ in (4.3) and (4.4). Consequently, we do not get the solution to fuzzy equation as a membership function; that is, we cannot determine $C_\infty$. Therefore, we need to reformulate the definition of "solution" in our setting.

Under the assumption that in (4.2) $C_n, C_0, C_1, a, b$, and $m$ are fuzzy numbers, we define $z^*$ to be a solution with belief $\alpha$ if $C_\infty(z^*) = \alpha$ and $[C_\infty]_\alpha = [L_{\infty,\alpha}, R_{\infty,\alpha}]$, where $L_{\infty,\alpha}$ and $R_{\infty,\alpha}$ is the solution to the system in (4.5). Now the expression $Z_n = P^{-1}Y_n$ implies that

$$Z_{1,n+1} = \frac{1}{2}\frac{\sqrt{a_{l,\alpha}b_{l,\alpha}}}{\sqrt{a_{r,\alpha}b_{r,\alpha}}}L_{n,\alpha} + \frac{1}{2}R_{n,\alpha}. \tag{4.11}$$

Note that if in (4.11), the quantity $\sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}} = 1$ provided that the product $ab$ in (4.1) is a crisp number, then $Z_{1,n+1}$ is the midpoint of the interval $[L_{n,\alpha}, R_{n,\alpha}]$. In this case, for a fixed $\alpha$-level, $Z_{1,n+1}$ converges to $Z_{1,\infty}$, where $Z_{1,\infty}$ is the solution of (4.2), where the belief is $\geq \alpha$. In fact, if $C_n$ is "near symmetric" around the solution with belief level $\alpha = 1$ (i.e., the solution to the classical equation), then $Z_{1,\infty}$ is near the solution that has maximum belief. The smaller the quantity $\sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}}$ is, the more "uncertain" we are about the product $ab$. We would like $Z_{1,\infty}$ to be a solution of (4.2) with belief $\geq \alpha$. Then we would like for $n$ large enough to have

$$L_{n,\alpha} \leq \frac{1}{2}\frac{\sqrt{a_{l,\alpha}b_{l,\alpha}}}{\sqrt{a_{r,\alpha}b_{r,\alpha}}}L_{n,\alpha} + \frac{1}{2}R_{n,\alpha} \leq R_{n,\alpha}$$

or

$$2 - \frac{R_{n,\alpha}}{L_{n,\alpha}} \leq \frac{\sqrt{a_{l,\alpha}b_{l,\alpha}}}{\sqrt{a_{r,\alpha}b_{r,\alpha}}}L_{n,\alpha} \leq \frac{R_{n,\alpha}}{L_{n,\alpha}}.$$

Note that $R_{n,\alpha}/L_{n,\alpha} \geq 1$ and the fact $\sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}} \leq 1$ imply that $\sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}} \leq R_{n,\alpha}/L_{n,\alpha}$ is true. Thus $Z_{1,\infty}$ will be a solution of (4.2) with degree at least $\alpha$ provided $ab$ is not "too uncertain", that is, $(\sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}})L_{n,\alpha}$ is not too small.

Suppose, from physiological considerations, we decide that eventually $R_{n,\alpha} \geq R$ and $L_{n,\alpha} \leq L$. Then we have a solution of (4.2) with a belief $\geq \alpha$ provided $2 - (R/L) \leq \sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}}$. Thus starting at a fixed level of belief $\alpha$, we obtain a number $Z_{1,\infty}$ which carries a belief of at least $\alpha$ to be the solution to (4.2) provided the product $ab$ is not too uncertain relative to the uncertainty surrounding $C_n$. In fact, this last condition is expressed by $2 - (R/L) \leq \sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}}$. If $ab$ is a crisp number, $\sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}} = 1$, we obtain a solution that for a "symmetric" $C_n$ has near optimal belief. The conditions $R_{n,\alpha} \geq R$, $L_{n,\alpha} \leq L$ and

$$\frac{\sqrt{a_{l,\alpha}b_{l,\alpha}}}{\sqrt{a_{r,\alpha}b_{r,\alpha}}} \geq 2 - \frac{R}{L} \geq 2 - \frac{R_{n,\alpha}}{L_{n,\alpha}}$$

intuitively imply, respectively, that $ab$ is relatively narrow with respect to $C_n$ for large $n$.

We may summarize the above discussion in the following result.

THEOREM 4.1. *Let $\alpha$ represent a fixed level. Then $Z_{1,\infty}$ is a solution of the fuzzy equation (4.2) with belief at least $\alpha$ provided that $ab$ is not too uncertain relative to the uncertainty surrounding $C_n$, that is, $2 - (R/L) \leq \sqrt{a_{l,\alpha}b_{l,\alpha}}/\sqrt{a_{r,\alpha}b_{r,\alpha}}$, where $R$ and $L$ are such that eventually $R_{n,\alpha} \geq R$ and $L_{n,\alpha} \leq L$.*

## REFERENCES

1. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
2. L. Edelstein-Keshet, *Mathematical Models in Biology*, McGraw-Hill, New York, (1987).
3. D.S. Jones and B.D. Sleeman, *Differential Equations and Mathematical Biology*, George Allen and Unwin, London, (1983).
4. W.G. Kelley and A. Peterson, *Difference Equations*, Academic Press, New York, (1990).
5. V. Lakshmikantham and D. Frigiante, *Theory of Difference Equations*, Academic Press, New York, (1990).
6. D. Dubois and H. Prade, Operations on fuzzy numbers, *Int. J. System Science* **9**, 613–626, (1978).
7. D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, (1980).
8. O. Kaleva, Fuzzy performance of a coherant system, *J. Math. Anal. Appl.* **117**, 234–246, (1986).
9. H.T. Nguyen, A note on the extension principle for fuzzy sets, *J. Math. Anal. Appl.* **64** (2), 369–380, (1978).
10. L.A. Zadeh, Fuzzy sets, information and control **8**, 338–353, (1965).
11. L.A. Zadeh, The concept of linguistic variables and its applications to approximate reasoning, Part I, *Inf. Science* **8**, 199–249, (1975).
12. L.A. Zadeh, The concept of linguistic variables and its applications to approximate reasoning, Part II, *Inf. Science* **8**, 301–357, (1975).
13. L.A. Zadeh, The concept of linguistic variables and its applications to approximate reasoning, Part III, *Inf. Science* **9**, 43–80, (1975).